On a Resolution of the Identity in Terms of Coherent States

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Abstract. Transformations of coherent states of the free particle by bounded and semibounded symmetry operators are considered. Resolution of the identity operator in terms of the transformed states is analyzed. A generalized identity resolution is formulated. Darboux transformation operators are analyzed as operators defined in a Hilbert space. Coherent states of multisoliton potentials are studied.

1. Introduction

The notion of coherent states is widely used in the modern quantum mechanics and mathematical physics [1]-[3]. The more typical properties of coherent states are summarized in the definition given by Klauder [4]. In our interpretation this definition looks like as follows.

Definition 1. *Every system of states described by vectors $|\psi_z\rangle$ is called the system of coherent states if the following conditions are fulfilled*:

(i) $|\psi_z\rangle \in H_0$ *where* H_0 *is a Hilbert space;*

(ii) $z \in \mathcal{D} \subset \mathbb{C}$;

(iii) $\mathcal{D}$ *is a domain endowed with a measure* $\mu(z,\bar{z})$, $z,\bar{z} \in \mathcal{D}$ *which is defined and finite on a class of Borel sets of* $\mathcal{D}$ *and guaranties the following resolution of the identity operator* $\mathbb{I}$ *on* H_0:

$$\int_{\mathcal{D}} d\mu |\psi_z\rangle\langle\psi_z| = \mathbb{I}; \tag{1}$$

(iv) $\forall z \in \mathcal{D}$, $|\psi_z\rangle$ *belong to a domain of definition of a Hamiltonian* h_0 *on* H_0 *and are solutions to the Schrödinger equation*

$$(i\partial_t - h_0)|\psi_z\rangle = 0. \tag{2}$$

Remark 1. *The integral in equation* (1) *should be understood in a week sense. This means that if* $|\psi_n\rangle$ *is an orthonormal basis in* H_0 *then this equation is equivalent to* $\int_{\mathcal{D}} d\mu \langle\psi_k|\psi_z\rangle \langle\psi_z|\psi_n\rangle = \delta_{nk}$. *If* $|\xi_n\rangle$ *is a Riesz basis in* H_0 (*i.e. the basis equivalent to orthonormal, see e. g.* [5]) *then formala* (1) *is equivalent to* $\int_{\mathcal{D}} d\mu \langle\xi_k|\xi_z\rangle\langle\xi_z|\xi_n\rangle = \langle\xi_k|\xi_n\rangle$.

Remark 2. *In general,* $\mathcal{D}$ *is a domain in* $\mathbb{C}^n$. *In this letter we will restrict ourselves by the case* $\mathcal{D} = \mathbb{C}$. *In this case we will omit the domain of integration in the integrals.*

Remark 3. *We introduce the property* (iv) *to satisfy the condition of "temporal stability" formulated in* [4].

The condition (iii) is one of the most remarcable properties of coherent states widely used in mathematical physics, quantum optics, group theory, and in other fields of physics and mathematics. For instance, it plays an important role in the Berezin quantization scheme [6], in the analysis of growth of holomorphic in $\mathcal{D}$ functions [7], in a general theory of phase space quasiprobability distributions [8], and in quantum state engineering [9].

In this letter we will demonstrate the insufficiency of this definition. In particular, we will construct a system of the vectors which satisfy all the conditions of the Definition 1 except for the condition (iii). We will show that this system satisfies a more general condition. In this respect we propose a generalization of the Definition 1.

Our example is related with the problem of a transformation of the coherent states. Let for a quantum system called initial system we know the coherent states $|\psi_z\rangle$ in the sense of the Definition 1. We suppose that this system has a nontrivial symmetry operator g_0 defined as usually as an operator that transforms every solution of the Schrödinger equation (2) into another solution of the same equation. Let $|\psi_z\rangle$ belong to the domain of definition of g_0. Consider the vectors $|\varphi_z\rangle = g_0|\psi_z\rangle$, $z \in \mathcal{D}$. It is clear that all the conditions of the Definition 1 except may be for the condition (iii) are fulfilled. Problems that may be raised in this respect are the following: (a) To describe the properties of g_0 in order that it produces the coherent states in the sense of the Definition 1; (b) To modify the condition (iii) when $|\varphi_z\rangle$ do not satisfy the property (iii); (c) To describe the properties of g_0 in order that it produces the coherent states in the sense of the modified definition.

In this letter we give two examples of transformations. The first transformation is bounded and does not violate the property (iii). The states obtained with the help of the second transformation which is unbounded but closed satisfy a more general condition then that given by the equation (1). We will show that the integral in the equation (1) should be replaced by a functional defined over the set of the finite holomorphic in $\mathcal{D}$ functions. So, in this letter we will concentrate our attention only on the problem (b) raised above.

Finally we apply obtained results to coherent states of multisoliton potentials. For the case of the one soliton potential such states have been first introduced in [10].

2. Coherent states of the free particle

The nonrelativistic free particle is the system very suitable for demonstrating various aspects of quantum mechanics. This is due in particular to the fact that the Schrödinger equation for this system has the more rich symmetry algebra. Moreover, the system of coherent states (in the sense of the Definition 1) is known for it [2]. Another important property of the free particle Schrödinger equation that we will use in this letter consists in the fact that this equation is the basis one for obtaining the reflectionless potentials with discrete energy levels disposed in the desired manner (so called multisoliton potentials, see e.g. [11]).

In this section we review briefly the well known constructions related to the Hilbert space of the states of the free particle that we will need further.

Orthonormal set of solutions $\psi_n(x,t)$ of the Schrödinger equation for the free particle is well known [12] and we do not cite it. We will denote by $\operatorname{span}\{\psi_n(x,t)\}$ the lineal (i.e. the space of the finite linear combinations) of the elements $\psi_n(x,t)$, $n = 0, 1, 2, \ldots$. This lineal is an everywhere dense set in the space $L^2(\mathbb{R})$ of the functions square integrable on $\mathbb{R}$ with respect to the Lebesgue measure. The basis $\psi_n(x,t)$ has the lowering a and raising a^+ operators, $a\psi_n(x,t) = \sqrt{n}\psi_{n-1}(x,t)$, $a\psi_0(x,t) = 0$, $a^+\psi_n(x,t) = \sqrt{n+1}\psi_{n+1}(x,t)$. The momentum operator p_x is expressed in terms of a and a^+ as follows: $p_x = -i\partial_x = -(a+a^+)/2$. The free particle Hamiltomian is $h_0 = -\partial_x^2 = p_x^2$.

Let us associate with the functions $\psi_n(x,t) = \langle x|\psi_n\rangle$ the elements $|\psi_n\rangle$ of an abstract vector space $\mathcal{L}_0 = \operatorname{span}\{|\psi_n\rangle\}$. Define the action of the linear raising a and lowering a^+ operators on the basis elements $|\psi_n\rangle$ by the same relations: $a|\psi_n\rangle = \sqrt{n}|\psi_{n-1}\rangle$, $a|\psi_0\rangle = 0$, $a^+|\psi_n\rangle = \sqrt{n+1}|\psi_{n+1}\rangle$. Since a and a^+ are supposed to be linear, their action is defined

for every $|\psi\rangle \in \mathcal{L}_0$ and $a|\psi\rangle \in \mathcal{L}_0$, $a^+|\psi\rangle \in \mathcal{L}_0$. Moreover, $p_x|\psi\rangle \in \mathcal{L}_0$, $h_0|\psi\rangle \in \mathcal{L}_0$, $\forall|\psi\rangle \in \mathcal{L}_0$ where $p_x = -(a+a^+)/2$, $h_0 = p_x^2$.

Let us define the scalar product $\langle\cdot|\cdot\rangle$ in $\mathcal{L}_0$, with the help of the coordinate representation $\psi(x,t) = \langle x|\psi\rangle$ of the vectors $|\psi\rangle$ and by using the ordinary Lebesgue integral.

Denote by H_0 the completion of $\mathcal{L}_0$, $H_0 = \bar{\mathcal{L}}_0$, with respect to the norm generated by this scalar product. It is well known that p_x and h_0 are essentially self adjoint operators in $L^2(\mathbb{R})$ with the well defined domains of definitions. In this context we will consider the closures $\bar{p}_x$ and $\bar{h}_0$ as the unique self adjoint extensions of the operators p_x and h_0 initially defined on $\mathcal{L}_0$. The operator $\bar{h}_0$ is bounded from below. We will denote $D_{h_0} \subset H_0$ its domain of definition which is dense in H_0. In this construction the evolution parameter t (time) is involved in every element $|\psi\rangle \in H_0$. Since a one parametric group of evolution operators U_t is continuous with respect to t and uniquely defined by the Schr¨dinger equation, the derivative of $|\psi\rangle$ with respect to t exists $\forall|\psi\rangle \in D_{h_0}$ [13] and every $|\psi\rangle \in D_{h_0}$ satisfies the Schrödinger equation.

The operators a and a^+ initially defined on $\mathcal{L}_0$ may be extended to a domain $D \supset \mathcal{L}_0$ common to both operators. The sum $a+a^+$, the products aa^+ and a^+a are defined $\forall|\psi\rangle \in D$. Moreover, $\langle\psi_b|a\psi_{b'}\rangle = \langle a^+\psi_b|\psi_{b'}\rangle$, $\forall|\psi_{b,b'}\rangle \in D$.

The basis vectors $|\psi_n\rangle$ are the eigenvectors of the operator $g_{00} = aa^+$, $g_{00}|\psi_n\rangle = (n+1)|\psi_n\rangle$. This operator is symmetric in D and bounded from below. The closure $\bar{g}_{00}$ of g_{00} is the unique self adjoint extension of this operator and it is defined $\forall|\psi\rangle \in D$. The operator $K = \bar{g}_{00}^{-1}$ is a Hilbert-Schmidt operator and it may be chosen to equip the Hilbert space H_0 by the spaces H_+ and H_-, $H_+ \subset H_0 \subset H_-$ where H_+ is dense in H_0 and H_- is the space of functionals off H_+. The operator K may be restricted to H_+. The operator K^+ conjugate to K has in this case a natural extension to the space H_-. The operator K defines an isometry $H_0 \to H_+$ and K^+ an isometry $H_- \to H_0$ (see e.g. [13]).

Every self adjoint in H_0 operator A has a complete system of the generalized eigenvectors $|\psi_\lambda\rangle$ which are the functionals from H_-. They are defined with the help of the measure $\sigma(\lambda) = \langle e|P_\lambda e\rangle$ where P_λ is the spectral function of A and $|e\rangle$ is some element from H_0 as follows: $|\psi_\lambda\rangle = \frac{d|P_\lambda e\rangle}{d\sigma(\lambda)}$, $A|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle$ [14]. The completeness of the system $\{|\psi_\lambda\rangle\}$ means that the Fourier transform $\psi(\lambda) = \langle\psi_\lambda|\psi\rangle$ of an element $|\psi\rangle \in H_0$ belongs to the space $L^2(d\sigma)$ of the functions square integrable with respect to the measure $\sigma(\lambda)$ and the Parseval

equality is valid $\langle\psi|\psi\rangle = \int\langle\psi|\psi_\lambda\rangle\langle\psi_\lambda|\psi\rangle d\sigma(\lambda)$ for all $|\psi\rangle \in H_0$. The inverse transform is written as $|\psi\rangle = \int d\sigma(\lambda)\psi(\lambda)|\psi_\lambda\rangle$. This equality should be understood in the week sense, i.e. $\forall|h\rangle \in H_+$ we have $\langle h|\psi\rangle = \int d\sigma(\lambda)\psi(\lambda)\bar{h}(\lambda)$. The latter relations may be summarized in the equation

$$\int d\sigma(\lambda)|\psi_\lambda\rangle\langle\psi_\lambda| = \mathbb{I}$$

that should be understood in the week sense.

Operator $\bar{p}_x$ has only a continuous spectrum. Let $|\psi_p\rangle$, $p \in \mathbb{R}$ be its generalized eigenvectors. Then their completeness and orthonormality may be written in the form

$$\langle\psi_q|\psi_p\rangle = \delta(q-p), \quad \int dp|\psi_p\rangle\langle\psi_p| = \mathbb{I}.$$

Coherent states $|\psi_z\rangle$ of the free particle may be obtained by the action on $|\psi_0\rangle$ by the displacement operator [3]

$$|\psi_z\rangle = \exp(za^+ - \bar{z}a)|\psi_0\rangle, \quad z \in \mathbb{C}.$$

These vectors are the eigenvectors of the lowering operator a, $a|\psi_z\rangle = z|\psi_z\rangle$, $z \in \mathbb{C}$. The Fourier expansion of $|\psi_z\rangle$ in terms of the basis $|\psi_n\rangle$ looks like as follows:

$$|\psi_z\rangle = \Phi\sum_n a_n z^n|\psi_n\rangle, \quad \Phi = \Phi(z,\bar{z}) = \exp(-z\bar{z}/2), \quad a_n = (n!)^{-1/2}.$$

We denote by $\bar{z}$ the value complex conjugate to z. The functions $\psi_z(x,t) = \langle x|\psi_z\rangle$ and $\psi_z(p,t) = \langle\psi_p|\psi_z\rangle$ are well known [2]. The vectors $|\psi_z\rangle$ are the coherent states in the sense of the Definition 1. The measure $d\mu$ in the equation (1) is equal to $d\mu = dxdy/\pi$, $z = x+iy$.

Since $|\psi_p\rangle$ is the basis in H_0 the equation (1) is equivalent to

$$\int d\mu\langle\psi_p|\psi_z\rangle\langle\psi_z|\psi_q\rangle = \delta(p-q). \tag{3}$$

3. Transformation of the coherent states by symmetry operators

Let us consider a linear symmetry operator g_0 initially defined on $\mathcal{L}_0$ with the help of an Hermitian matrix $S = \|S_{nk}\|$, $S_{nk} = \bar{S}_{kn}$, $g_0|\psi_n\rangle = \sum_k S_{kn}|\psi_k\rangle$. We will suppose that every row (and column consequently) of the matrix S contains a finite number of non zero elements. In this case the operator g_0 is symmetric in $\mathcal{L}_0$ and maps $\mathcal{L}_0 \to \mathcal{L}_0$. Moreover, we

will suppose that g_0 is bounded from below, positive definite, and essentially self adjoint in H_0 so that $\bar{g}_0 = \bar{g}_0^+$. Let $D_0(\subset H_0)$ be the domain of definition of $\bar{g}_0$.

Under these assumptions the operator $\bar{g}_0^{-1}$ is uniquely defined and bounded in H_0. Its domain of definition is the whole H_0.

The operators $\bar{g}_0^{\pm 1/2}$ such that $\bar{g}_0^{\pm 1/2}\bar{g}_0^{\pm 1/2} = \bar{g}_0^{\pm 1}$ are uniquely defined on H_0 as well. The domain of definition of $\bar{g}_0^{-1/2}$ is the whole H_0. Denote $D_0'(\supset D_0)$ the domain of definition of $\bar{g}_0^{1/2}$. It may be analyzed with the help of Friedrichs extension (see e.g. [15]) of g_0 up to $\bar{g}_0$. We notice that $|\psi_n\rangle \in D_0 \subset D_0'$

Lemma 1. *The systems* $\{|\rho_n\rangle\}$, $|\rho_n\rangle = \bar{g}_0^{1/2}|\psi_n\rangle$ *and* $\{|\xi_n\rangle\}$, $|\xi_n\rangle = \bar{g}_0^{-1/2}|\psi_n\rangle$ *are biorthogonal Riesz basiss in* H_0

We will not dwell on the proof of this lemma. We note only that

$$\langle\rho_n|\rho_k\rangle = S_{nk},\ \langle\xi_n|\xi_k\rangle = S^{-1}_{nk},\ \sum_k S_{nk}S^{-1}_{kj} = \delta_{nj},\ \langle\xi_n|\rho_k\rangle = \delta_{nk}.$$

Corollary 1. *The equation* (1) *may be rewritten both in terms of the basis* $|\rho_n\rangle$

$$\int d\mu\langle\rho_n|\psi_z\rangle\langle\psi_z|\rho_k\rangle = S_{nk}$$

and in terms of the basis $|\xi_n\rangle$

$$\int d\mu\langle\xi_n|\psi_z\rangle\langle\psi_z|\xi_k\rangle = S^{-1}_{nk}.$$

The elements S_{nk} of the matrix S and the elements S^{-1}_{nk} of the matrix S^{-1} may be calculated with the help of the generalized eigenvectors $|\psi_\lambda\rangle$ of the operator $\bar{g}_0$, $\bar{g}_0|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle$

$$S^{\gamma}_{nk} = \int d\sigma(\lambda)\lambda^{\gamma}\bar{\psi}^{(\gamma)}_n(\lambda)\psi^{(\gamma)}_k(\lambda), \quad \gamma = \pm 1,$$
$$\psi^{(1)}_n(\lambda) = \langle\psi_\lambda|\rho_n\rangle, \quad \psi^{(-1)}_n(\lambda) = \langle\psi_\lambda|\xi_n\rangle$$

where $d\sigma(\lambda)$ is the measure that guaranties the spectral resolution of $\bar{g}_0$.

It is not difficult to see that if $f(x)$ is some positive polynomial, $f(x) > 0$, $\forall x \in \mathbb{R}$ and $g_0 = f(p_x)$ then all the above assumptions imposed on g_0 are fulfilled.

Theorem 1. *If* $g_0 = f(p_x)$ *where* $f(x)$ *is some positive polynomial in* $x(\in \mathbb{R})$ *then the vectors* $|\xi_z\rangle = \bar{g}_0^{-1/2}|\psi_z\rangle = \Phi\sum_n a_n z^n|\xi_n\rangle$ *describe coherent states in the sense of the Definition 1.*

Proof. It is obvious that it is sufficient to establish the resolution of the identity operator. Our proof is constructive and we will only sketch it.

Let us suppose that the measure $\mu_\xi = \mu_\xi(z, \bar{z})$ that realizes the resolution of the identity in terms of the vectors $|\xi_z\rangle$ exists and try to find it. We will see that it is possible if the measure is such that $d\mu_\xi = \omega_\xi(x)dxdy$.

To find the density $\omega_\xi(x)$ we use the generalized eigenvectors $|\psi_p\rangle$ of the operator $\bar{p}_x$ which are the eigenvectors of $\bar{g}_0$ as well. Then using the expression

$$\langle\psi_p|\psi_z\rangle = (2/\pi)^{1/4}\Phi\psi_p(z), \quad \psi_p(z) = \exp(-p^2 + 2zp - z^2/2), \quad z = x + iy,$$

the form (3) of the formula (1), and integrating with respect to the variable y we arrive at the equation for $\omega_\xi(x)$

$$\int dx\omega_\xi(x)F_p(x) = (2\pi)^{-1/2}f(p)\exp(2p^2), \quad F_p(x) = \exp(4px - 2x^2). \qquad (4)$$

It is clearly seen from this relation that the smooth function $\omega_\xi(x)$ is a polynomial in x completely defined by the coefficients of the polynomial $f(p)$. This proves the assertion. (Q.E.D.)

Corollary 2. *The measure $d\mu_\xi$ is a solution to the follwing problem of moments on the complex plane*:

$$a_n a_k \int d\mu_\xi |\Phi|^2 \bar{z}^n z^k = S_{nk}.$$

This assertion follows immediately from the resolution of the identity in terms of the vectors $|\xi_z\rangle$ and the property $\langle\rho_n|\xi_z\rangle = \Phi a_n z^n$ (see Lemma 1).

Remark 4. *Given the identity resolution we may construct a holomorphic representation of the space H_0 and the operators on it. We do not dwell on these constructions.*

Let us consider the vectors $|\rho_z\rangle = \Phi\sum_n a_n z^n|\rho_n\rangle$. The function $A_n(z) = \sum_k a_k z^k S_{nk}$ is a polynomial in z since the sum is finite. It is not difficult to see that when $g_0 = f(p_x)$ and $f(x)$ is a polynomial of order $2N$ then the number $\max_k S_{nk}$ have the following asymptotic behavior: $\max_k S_{nk} \to Cn^N$ when $n \to \infty$ where C is a constant. As a result the series $\sum_n a_n A_n(z)\bar{z}^n$ converges $\forall z \in \mathbb{C}$. This means that the vector $|\rho_z\rangle$ has a finite norm and consequently $|\rho_z\rangle \in H_0$. It follows that $\bar{g}_0^{-1/2}|\rho_z\rangle = \Phi\sum_n a_n z^n|\psi_n\rangle = |\psi_z\rangle \in D_0'$ and

$\bar{g}_0^{1/2}|\psi_z\rangle = |\rho_z\rangle$. The natural question that arises in this respect is the following: whether the states $|\rho_z\rangle$ may be considered as coherent states.

Let us suppose that the measure $\mu_\rho = \mu_\rho(z,\bar{z})$ that realizes the resolution of the identity operator in terms of the vectors $|\rho_z\rangle$ exists and try to find it. Taking into account the above considerations we will suppose that μ_ρ is such that $d\mu_\rho = d\nu_\rho(x)dy$. The equation (4) takes in this case the form:

$$\int d\nu_\rho(x)F_p(x) = (2\pi)^{-1/2}\exp(2p^2)/f(p) \tag{5}$$

We have not succeed to solve this equation in the ordinary functions but we have found its solution as a generalized function. We notice that $|F_p(x+iy)| \le \exp(-dx^2+by^2), \quad 2 \le d \le b$. It follows that $F_p(x) \in S_{1/2}^{1/2}$, where the space $S_{1/2}^{1/2}$ is defined as the space of entire functions F such that $|F(x+iy)| \le \exp(-dx^2+by^2)$, $0 < d \le b$ [16]. We may try to find ν_ρ as a functional off the space $S_{1/2}^{1/2}$.

It is well known [16] that a positive definite generalized function ν over the space $S_{1/2}^{1/2}$ may have an integral representation in the space of the Fourier transforms.

Let $\tilde{F}_p(t) = \sqrt{\pi/2}\exp(2p^2+ipt-t^2/8)$ be the Fourier transform of the function $F_p(x)$. We will understand the integral in the left hand side of the equation (5) as a generalized function over the space $S_{1/2}^{1/2}$ defined by the measure $\tilde{\nu}_\rho(t)$ in the space of the Fourier transforms

$$\int d\nu_\rho(x)F_p(x) = \int d\tilde{\nu}_\rho(t)\tilde{F}_p(t). \tag{6}$$

The equation (5) results then in the equation for $\tilde{\nu}_\rho(t)$

$$\pi\int d\tilde{\nu}_\rho(t)\exp(-t^2/8+ipt) = \frac{1}{f(p)}. \tag{7}$$

We will give the solution to this equation for the particular case of the function $f(x)$ that we need further. Let $f(x)$ be a polynomial of order $2N$ and the zeros of $f(x)$ be purely imaginary. Every such a polynomial may be presented in the form: $f(x) = f_0(x) = A_0\prod_{k=1}^{N}(x^2+\alpha_k^2)$, $\alpha_k > 0$. The value of the coefficient A_0 is without importance for our purpose and we put $A_0 = 1$. Then the function $1/f(x)$ may be presented in the form:

$$\frac{1}{f(x)} = \sum_{k=1}^{N}\frac{A_k}{x^2+\alpha_k^2}, \quad A_k = \left[\frac{df(x)}{d(x^2)}\right]_{x^2=-\alpha_k^2}.$$

It can be seen now from (7) that if $d\tilde{\nu}_\rho(t) = \tilde{\omega}_\rho(t)dt$ then for the density $\tilde{\omega}_\rho(t)$ we have the expression

$$\tilde{\omega}_\rho(t) = (2\pi)^{-1} \sum_{k=1}^{N} \frac{A_k}{\alpha_k} \exp(-\alpha_k |t| + t^2/8). \tag{8}$$

It is necessary to note that the integral in the right hand side of (6) converges not for all functions $F(x) \in S_{1/2}^{1/2}$. It is easy to see that the convergence condition of this integral translates into the condition of the decreasing of the functions $F(x)$ when $|x| \to \infty$. We should take only such functions $F(x) \in S_{1/2}^{1/2}$ that the following inequality takes place: $|F(x)| \geq \exp(-2x^2 - Ax)$ where $A \geq 0$ is some positive constant own to the function $F(x)$. Let us denote by $\overset{\circ}{S}{}_{1/2}^{1/2}$ all the functions from $S_{1/2}^{1/2}$ satisfying this condition. It is apparent that $\overset{\circ}{S}{}_{1/2}^{1/2}$ is a linear space.

We see hence that the integral in the left hand side of the equation (3) should be understood as an Hermitian, linear on the second argument and antilinear on the first argument, continuous on both arguments, bounded, positive definite functional ω_ρ acting on the functions

$$\langle \psi_p | \rho_z \rangle = f^{1/2}(p) \langle \psi_p | \psi_z \rangle = f^{1/2}(p) \Phi \psi_p(z), \;\; \psi_p(z) = \exp(-p^2 + 2zp - z^2/2), \;\; p \in \mathbb{R}$$

as follows

$$\omega_\rho(\langle \psi_q | \rho_z \rangle, \langle \rho_z | \psi_p \rangle) = \delta(q - p).$$

This functional is defined with the help of the Fourier transform $\tilde{F}_p(t)$ of the function $F_p(x)$ defined by the relation

$$\int_{-\infty}^{\infty} dy F_{pq}(z, \bar{z}) = \delta(p - q) F_p(x), \quad F_{pq}(z, \bar{z}) = |\Phi|^2 \bar{\psi}_q(z) \psi_p(z).$$

It is not difficult to see that the function

$$F_{a,b}(x) = \int_{-\infty}^{\infty} dy \langle \rho_a | \rho_z \rangle \langle \rho_z | \rho_b \rangle, \quad z = x + iy,$$

belongs to the space $\overset{\circ}{S}{}_{1/2}^{1/2}$ when $|\rho_{a,b}\rangle \in \mathcal{L}_\rho$. The same is true when we take $|\xi_{a,b}\rangle \in \mathcal{L}_\xi$ instead of $|\rho_{a,b}\rangle$. By using the biorthogonality of the systems $\{|\xi_n\rangle\}$ and $\{|\rho_n\rangle\}$ (see Lemma 1) we derive that $\langle \xi_n | \rho_z \rangle = \Phi a_n z^n$. It follows that the functional ω_ρ gives a solution to the following problem of moments on the complex plane:

$$a_n a_k \omega_\rho(\Phi z^n, \Phi z^k) = S_{nk}^{-1}. \tag{9}$$

(Compare with the Corollary 2.)

The function Φ is defined by the initial coherent states and it does not depend on the transformation used. This means that the functional ω_ρ is really defined on the elements $a_n z^n$ which are the basis elements in the space $\mathcal{L}_\mu$ of the finite holomorphic functions. This functional has all the properties necessary to define a scalar product in $\mathcal{L}_\mu$. Therefore we may consider the completion of the lineal $\mathcal{L}_\mu$ with respect to the norm generated by this scalar product and obtain the Hilbert space $H_\mu = \bar{\mathcal{L}}_\mu$ of functions holomorphic in $\mathbb{C}$. (Compare with the Remark 4.)

It is now clear how we should adjust the Definition 1 to take in consideration the obtained results.

Definition 2. *Let $|\psi_n\rangle$ be such a basis in the Hilbert space H_0 that the vectors $|\psi_z\rangle$ have the form: $|\psi_z\rangle = \Phi \sum_n a_n z^n |\psi_n\rangle$ where $a_n > 0$ are some numbers and $\Phi = \Phi(z,\bar{z})$ is some real valued function. By coherent states we shall mean the states described by the vectors $|\psi_z\rangle$ which satisfy the Definition 1 where the property* (iii) *is replaced by* (iii): *in the space of the finite holomorphic functions there exists a functional ω which is Hermitian, linear in the second argument and antilinear in the first argument, continuous in both arguments, bounded, positive definite, and such that the following resolution of the identity operator acting in H_0 is valid*

$$\omega(|\psi_z\rangle\Phi, \Phi\langle\psi_z|) = \mathbb{I}$$

where the equality is understood in the week sense.

We summarize the above results in the following

Theorem 2. *The states $|\rho_z\rangle = \bar{g}_0^{-1/2}|\psi_z\rangle = \sum_n a_n z^n |\rho_n\rangle$ are coherent states in the sense of the Definition 2. The functional $\omega = \omega_\rho$ which realizes the resolution of the identity in terms of the vectors $|\rho_z\rangle$ gives the solution to the problem of moments* (9).

4. Applications. Coherent states of multisoliton potentials

It is well known (see e.g. [11]) that the soliton potentials may be obtained by Darboux transformations of the free particle Schrödinger equation. The Darboux transformation operator, L, that transforms the solutions of the Schrödinger equation with zero potential to the solutions of the same equation with multisoliton potential is well known as well [11].

Let h_1 be the Hamiltonian of a multisoliton potential having N discrete spectrum levels $E_{-i} = -\alpha_{-i}^2$, $\alpha_i > 0$ with the eigenfunctions $\varphi_{-i}(x,t)$, $i = 1,\ldots,N$. If we denote L^+ the operator formally adjoint to L then $\ker L^+ = \mathrm{span}\,\{\varphi_{-i}\}$. The operators L and L^+ have the remarkable factorization property [17]: $L^+L = f(h_0) = g_0$, $LL^+ = f(h_1) = g_1$ where $f(x) = \prod_{k=1}^{N}(x+\alpha_i^2)$. It should be noted that $h_0 = p_x^2$ and $f(p_x^2) = f_0(p_x)$ where $f_0(x)$ is the polynomial introduced above. The functions $\varphi_n(x,t) = L\psi_n(x,t)$ are orthogonal to the space $\ker L^+$. Therefor we may consider the orthogonal decomposition $L^2(\mathbb{R}) = L_0^2 \oplus L_1^2$ where L_0^2 is the closure of the space $\ker L^+$ and L_1^2 is the closure of span $\{\varphi_n(x,t)\}$, $n = 0,1,2,\ldots$. In what follows we will not consider the space L_0^2 and concentrate our attention only on the space L_1^2.

Operators L, L^+, g_0 and g_1 participate in the following intertwining relations: $Lg_0 = g_1L$, $g_0L^+ = L^+g_1$ which hold in the lineal span $\{\psi_n(x,t)\}$ and span $\{\varphi_n(x,t)\}$ respectively. Therefor if $g_0\psi_n(x,t) = \sum_k S_{kn}\psi_k(x,t)$ then $g_1\varphi_n(x,t) = \sum_k S_{kn}\varphi_k(x,t)$ (the sums are finite). We notice as well that $\bar{g}_0$ and $\bar{g}_1$ are the unique self adjoint extensions of g_0 and g_1 respectively.

Let $\psi_p(x,t)$ be the generalized eigenfunctions of $\bar{h}_0$

$$\bar{h}_0\psi_p(x,t) = p^2\psi_p(x,t), \quad \langle\psi_p(x,t)|\psi_q(x,t)\rangle = \delta(p-q), \quad p,q \in \mathbb{R}.$$

The functions

$$\varphi_p(x,t) = N_p^{-1}L\psi_p(x,t), \quad N_p^2 = f(p^2), \quad \langle\varphi_p(x,t)|\varphi_q(x,t)\rangle = \delta(p-q)$$

are the generalized eigenfunctions of $\bar{h}_1$, $\bar{h}_1\varphi_p(x,t) = p^2\varphi_p(x,t)$.

The operator L^+ realizes the transformation in the inverse direction, $L^+\varphi_p(x,t) = N_p\psi_p(x,t)$. The functions $\psi_p(x,t)$ are the generalized eigenfunctions of the operator $\bar{g}_0 = f(\bar{h}_0)$, $\bar{g}_0\psi_p(x,t) = f(p^2)\psi_p(x,t)$ and the functions $\varphi_p(x,t)$ are the similar ones for $\bar{g}_1 = f(\bar{h}_1)$, $\bar{g}_1\varphi_p(x,t) = f(p^2)\varphi_p(x,t)$.

In this section we will give rigorous constructions related with the multisoliton potentials and establish the relationship between the Darboux transformation operator L and the spectral resolutions of the operators $\bar{g}_0$ and $\bar{g}_1$. A polar factorization of the operator L will be derived. Then we will introduce two kinds of coherent states for multisoliton potentials. The first states are the coherent ones in the sense of the Definition 1 and the second states are the similar ones but in the sense of the Definition 2.

Let us associate with the functions $\varphi_n(x,t)$ the vectors $|\varphi_n\rangle = L|\psi_n\rangle$. Denote by $\mathcal{L}_1 =$ span $\{|\varphi_n\rangle\}$, $n = 0, 1, 2, \ldots$. The operator L is supposed to be linear by definition. Therefor it is defined $\forall \psi \in \mathcal{L}_0$ and maps $\mathcal{L}_0$ onto $\mathcal{L}_1$. Let us define for every $|\varphi_n\rangle = L|\psi_n\rangle$, $n = 0, 1, 2, \ldots$ the linear operator L^+ by the relation $L^+|\varphi_n\rangle = g_0|\psi_n\rangle = \sum_k S_{kn}|\psi_k\rangle$ (the sum is finite). It is clear that L^+ is defined $\forall|\varphi\rangle \in \mathcal{L}_1$ and it maps $\mathcal{L}_1$ onto $\mathcal{L}_0$.

Let us define the scalar product on $\mathcal{L}_1$ by the equation

$$\langle\varphi_a|\varphi_b\rangle_1 \equiv \langle\psi_a|\bar{g}_0|\psi_b\rangle_0, \quad |\psi_{a,b}\rangle \in \mathcal{L}_0, \quad |\varphi_{a,b}\rangle = L|\psi_{a,b}\rangle \in \mathcal{L}_1.$$

Henceforth we label the scalar product in H_0 by the subscript 0. Let $H_1 = \bar{\mathcal{L}}_1$ be the completion of $\mathcal{L}_1$ with respect to the norm generated by this scalar product. It is necessary to note that the set $\{|\varphi_n\rangle\}$ is a basis in H_1, $\langle\varphi_k|\varphi_n\rangle_1 = S_{kn}$. We will show further that it is a Riesz basis and will find a basis biorthogonal with it.

Lemma 2. *The operator L has such an extension $\bar{L}$ that it domain of definition is D_0' and it domain of values is H_1.*

Proof. Let $|\psi\rangle = \sum_n c_n|\psi_n\rangle$ be a vector from D_0'. Then $\langle\bar{g}_0^{1/2}\psi|\bar{g}_0^{1/2}\psi\rangle_0 = \sum_{nk}\bar{c}_n c_k S_{nk} = c < \infty$. It follows that $\langle\varphi|\varphi\rangle_1 = c < \infty$ where $|\varphi\rangle = \sum_n c_n|\varphi_n\rangle$. So, we may put $\bar{L}|\psi\rangle = |\varphi\rangle = \sum_n c_n|\varphi_n\rangle$, $\forall|\psi\rangle \in D_0'$ as the definition of $\bar{L}$. It is evident that $\bar{L}$ is the extension of L since $\bar{L}|\psi_n\rangle = L|\psi_n\rangle = |\varphi_n\rangle$.

Consider an element $|\varphi\rangle \in H_1$, $|\varphi\rangle = \sum_n c_n|\varphi_n\rangle$, $\langle\varphi|\varphi\rangle_1 = \sum_{nk}\bar{c}_n c_k S_{nk} = c < \infty$. Consider now the vector $|\psi\rangle \in H_0$ defined by the same Fourier coefficients c_n with respect to the basis $|\psi_n\rangle$, $|\psi\rangle = \sum_n c_n|\psi_n\rangle$. The square of the norm of $\bar{g}_0^{1/2}|\psi\rangle$ equal $\sum_{nk}\bar{c}_n c_k S_{nk} = c$ is finite. It follows that $|\psi\rangle \in D_0'$ and by the definition of $\bar{L}$ we have $\bar{L}|\psi\rangle = |\varphi\rangle$. This proves the assertion. (Q.E.D.)

Let us define now for every $|\varphi\rangle = \bar{L}|\psi\rangle \in H_1$, such that $|\psi\rangle \in D_0(\subset D_0)$, the operator $\bar{L}^+$ by the formula $\bar{L}^+|\varphi\rangle = \bar{g}_0|\psi\rangle$. Let D_1 be it domain of definition. The domain D_1 consists of all $|\varphi\rangle \in H_1$ of the form $|\varphi\rangle = \bar{L}|\psi\rangle$ where $|\psi\rangle \in D_0$. It is obvious that D_1 is dence in H_1.

Lemma 3. $\bar{g}_0 = \bar{L}^+\bar{L}$.

Proof. Since $\forall|\varphi\rangle \in \mathcal{L}_0$ we have $\bar{L}^+\bar{L}|\varphi\rangle \in \mathcal{L}_0$ and $\bar{L}^+\bar{L}|\varphi\rangle = g_0|\varphi\rangle$, the operator $\bar{L}^+\bar{L}$ is the extension of g_0 and it has the domain of definition equal D_0. Taking into consideration

that g_0 has the unique extension $\bar{g}_0$ with D_0 as the domain of definition we get $\bar{g}_0 = \bar{L}^+\bar{L}$. (Q.E.D.)

Lemma 4. *$\bar{L}^+$ is adjoint to $\bar{L}$ with respect to the scalar products $\langle\cdot|\cdot\rangle_0$ and $\langle\cdot|\cdot\rangle_1$*

Proof. The assertion follows from the chain of equalities $\langle\bar{L}^+\varphi_a|\psi_b\rangle_0 = \langle\bar{g}_0\psi_a|\psi_b\rangle_0 = \langle\psi_a|\bar{g}_0\psi_b\rangle_0 = \langle\varphi_a|\varphi_b\rangle_1 = \langle\varphi_a|\bar{L}\psi_b\rangle_1$ where $|\psi_{a,b}\rangle \in D_0$, $|\varphi_{a,b}\rangle = \bar{L}|\psi_{a,b}\rangle \in D_1$. (Q.E.D.)

Lemma 5. $\bar{L} = \bar{L}^{++}$.

Proof. The assertion follows immediately from the equality $\bar{g}_0^+ = \bar{g}_0$ and Lemma 3. (Q.E.D.)

Corollary 3. *The operator $\bar{L}$ is closed.*

We will take without proving the following

Proposition 1. *There exists an equipment of H_1, $H_{1+} \subset H_1 \subset H_{1-}$ such that the set of functionals from H_{1-} of the form $|\varphi_p\rangle = N_p^{-1}\bar{L}|\psi_p\rangle$, $|\psi_p\rangle \in H_{1-}$, $N_p^2 = f(p^2)$, $p \in \mathbb{R}$, is orthonormal $\langle\varphi_q|\varphi_p\rangle_1 = \delta(q-p)$ and complete in H_1.*

We denote by the same symbol $\bar{L}$ the extension of $\bar{L}$ to the space H_-.

A similar statement has been first formulated by Krein [18] for a Sturm-Liouville problem. It really means that the Darboux transformation operator being applied to a complete set of vectors of the initial system gives once again a complete set of vectors (in an appropriate space) of the transformed system.

Let us denote $\bar{g}_1 = \bar{L}\bar{L}^+$. It follows from Lemma 5 that $\bar{g}_1^+ = \bar{g}_1$. Moreover, since $\bar{g}_0|\psi_p\rangle = f(p^2)|\psi_p\rangle$, $p \in \mathbb{R}$, we have $\bar{g}_1|\varphi_p\rangle = f(p^2)|\varphi_p\rangle$. This implies that if $\bar{g}_1 = f(\bar{h}_1)$ then $\bar{h}_1|\varphi_p\rangle = p^2|\varphi_p\rangle$ and the self-adjoint operator $\bar{h}_1$ is defined by the spectral resolution

$$\bar{h}_1 = \int dpp^2|\varphi_p\rangle\langle\varphi_p|$$

in a dense set of the space H_1. It is clear that $\bar{h}_1$ corresponds to the restriction of the multisoliton Hamiltonian to the space L_1^2.

If we define the linear symmetric operator g_1 on $\mathcal{L}_1$ by the formula $g_1|\varphi_n\rangle = \sum_k S_{kn}|\varphi_k\rangle$ (the sum is finite) then $\bar{g}_1$ is the unique self adjoint extension of g_1.

The basis $\{|\varphi_n\rangle\}$ is not orthogonal in H_1. We will construct now the basis $\{|\eta_n\rangle\}$ biorthogonal to $\{|\varphi_n\rangle\}$.

Since $\bar{g}_0$ is positive definite in H_0, the equation $\bar{L}^+|\varphi\rangle = 0$, $|\varphi\rangle \in D_1$ has the unique solution $|\varphi\rangle = 0$. It follows that $\bar{L}^+$ is invertible. Let us define $\forall|\varphi\rangle \in D_1$ the operator M by the relation $M\bar{L}^+|\varphi\rangle = |\varphi\rangle$. The vector $\bar{L}^+|\varphi\rangle = \bar{g}_1|\psi\rangle$ belongs to the space H_0 when $|\psi\rangle \in D_0$. Therefor the operator $M = (\bar{L}^+)^{-1}$ is defined on H_0. The restriction of M on the lineal $\mathcal{L}_0$ coincides with the integral transformation operator introduced in [19, 20].

Consider the vectors $|\eta_n\rangle = M|\psi_n\rangle$. It is clear that $\langle\eta_k|\varphi_n\rangle_1 = \langle\bar{L}\psi_k|M\psi_n\rangle_1 = \delta_{kn}$. Therefor the system $\{|\eta_n\rangle\}$ is biorthogonal to $\{|\varphi_n\rangle\}$.

Proposition 2. *Operator $U = \bar{L}\bar{g}_0^{-1/2}$ realizes the isometric mapping of the domain D_0' onto D_1. Operator $U^+ = U^{-1} = \bar{g}_0^{-1/2}\bar{L}^+$ realizes the inverse mapping. Operators U and U^+ have the following resolutions in terms of the generalized eigenvectors $|\psi_p\rangle$ and $|\varphi_p\rangle$:*

$$U = \int dp|\varphi_p\rangle\langle\psi_p|, \quad U^+ = \int dp|\psi_p\rangle\langle\varphi_p|. \tag{10}$$

Proof. Let $|\psi\rangle \in D_0'$. Then $\bar{g}_0^{-1/2}|\psi\rangle \in D_0$ since $\bar{g}_0\bar{g}_0^{-1/2}|\psi\rangle = \bar{g}_0^{1/2}|\psi\rangle$. Therefor $\bar{L}\bar{g}_0^{-1/2}|\psi\rangle \in D_1$ when $|\psi\rangle \in D_0$. If $|\varphi\rangle \in D_1$ then $\bar{L}^+|\varphi\rangle = |\psi\rangle \in H_0$ and $U^+|\varphi\rangle = \bar{g}_0^{-1/2}|\psi\rangle \in D_0'$. This means that the domain of values of U^+ is D_0' and it is defined on D_1. It is easy to check that U conserves the value of the norm $\langle\psi|\psi\rangle_0 = \langle\bar{L}\bar{g}_0^{-1/2}\psi|\bar{L}\bar{g}_0^{-1/2}\psi\rangle_1$. The first formula (10) follows from the spectral resolution for $\bar{g}_0^{-1/2}$. The second formula is the conjugation of the first. (Q.E.D.)

Corollary 4. *From* (10) *it follows the spectral representation for $\bar{L}$ and $\bar{L}^+$*

$$\bar{L} = \int dpN_p|\varphi_p\rangle\langle\psi_p|, \quad \bar{L}^+ = \int dpN_p|\psi_p\rangle\langle\varphi_p|$$

and the similar representation for M and M^+

$$M = \int dpN_p^{-1}|\varphi_p\rangle\langle\psi_p|, \quad M^+ = \int dpN_p^{-1}|\psi_p\rangle\langle\varphi_p|.$$

Operators M and M^+ are bounded and factorize the operators $\bar{g}_0^{-1}$ and $\bar{g}_1^{-1}$: $M^+M = \bar{g}_0^{-1}$, $MM^+ = \bar{g}_1^{-1}$.

Corollary 5. *The set $\{|\xi_n\rangle\}$, $|\xi_n\rangle = U|\psi_n\rangle = \bar{g}_1^{1/2}|\eta_n\rangle = \bar{g}_1^{-1/2}|\varphi_n\rangle$ is an orthonormal set in H_1 and $\{|\eta_n\rangle\}$, $\{|\varphi_n\rangle\}$ are biorthogonal Riesz basiss in H_1.*

Remark 5. *The representation $\bar{L} = U\bar{g}_0^{1/2}$ is a canonical representation of the closed operator $\bar{L}$ and $M = U\bar{g}_0^{-1/2}$ is the similar representation of the bounded operator M (see e.g.* [21]*). These representations are called polar factorisations as well* [22].

We have seen that $|\psi_z\rangle \in D_0'$. Therefor the action of $\bar{L}$ on $|\psi_z\rangle$ is defined. Consider the vectors

$$|\varphi_z\rangle = \bar{L}|\psi_z\rangle = \Phi\sum_n a_n z^n|\varphi_n\rangle, \quad |\eta_z\rangle = M|\psi_z\rangle = \Phi\sum_n a_n z^n|\eta_n\rangle.$$

Proposition 3. *The states associated with the vectors $|\eta_z\rangle$ are coherent states in the sense of the Definition 1. The vectors $|\varphi_z\rangle$ satisfy the Definition 2.*

Proof. We note first of all that all the conditions of the definitions 1 and 2 are fulfilled except may be for the property (iii) and (iii). If the measure $\mu_\eta = \mu_\eta(z,\bar{z})$ that realizes the resolution of the identity in terms of the vectors $|\eta_z\rangle$ exists then it gives the solution to the problem of moments

$$a_n a_k \int d\mu_\eta |\Phi|^2 z^n \bar{z}^k = \langle\varphi_n|\varphi_k\rangle_1 = S_{nk}.$$

According to the Corollary 3 we have $d\mu_\eta = d\mu_\xi$. Similarly, if the functional $\omega = \omega_\varphi$ that realizes the identity decomposition in terms of the vectors $|\varphi_z\rangle$ exists then it should give the solution to the problem

$$a_n a_k \omega_\varphi(\Phi z^n, \Phi z^k) = \langle\eta_n|\eta_k\rangle_1 = S_{nk}^{-1}$$

According to the Theorem 2 we state that it is the functional ω_ρ that solves this problem and $\omega_\varphi = \omega_\rho$. This proves the assertion. (Q.E.D.)

5. Conclusion

In this letter we have given a rigorous mathematical meaning to the idea first stated in [19, 20]. In short terms, we have formulated such a definition of coherent states that the states obtained by means of the Darboux transformation operator from coherent states of the free particle are coherent states of multisoliton potential. The main feature of our definition is the existence of the resolution of the identity operator which has a more general form then that ordinary used in other definitions. Nevertheless, our resolution of the identity permits one to construct the holomorphic representation of the Hilbert space of the states of a quantum

system and the operators on it. This open the door to introduce a phase space and classical observables as covariant Berezin symbols [6]. In this way it is possible to obtain a classical system the Berezin quantization of which gives the quantum system that we have started with. This program has been realized in [23] for the potential $V_0(x) = \gamma x^{-2}$. They have shown that at classical level the Darboux transformation reduces to such a transformation of Kähler potential and Poisson bracket that the equations of motion remain unchanged. The similar calculations are now possible for multisoliton potentials.

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